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ON THE TYPE AND COTYPE OF BANACH SPACES

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ABSTRACT

It is shown that, for $1 < p < \infty$, $p \neq 2$, the notions of equal-norm type p and equal-norm cotype p are, in general, strictly weaker than those of type p, respectively cotype p.

A Banach space X is said to be of *type p*, respectively of *cotype p*, if there exists a constant $M < \infty$ so that, for every choice of vectors $\{x_i\}_{i=1}^n$ in X, we have

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t)x_i \right\| dt \leqq M \left(\sum_{i=1}^n \| x_i \|^p \right)^{1/p},$$

respectively,

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t)x_i \right\| dt \geqq M^{-1} \left(\sum_{i=1}^n \| x_i \|^p \right)^{1/p},$$

where $\{r_i\}_{i=1}^\infty$ denote the Rademacher functions.

These notions are among the most interesting invariants of the isomorphic theory of Banach spaces and in several concrete situations it is very important to determine the type and the cotype of a given space. This matter can be settled easily for many classical spaces but, in general, it is quite difficult. A remarkable computation of the type occurs in [3]. In this paper, R. C. James proved that a suitable modification of some uniformly nonoctahedral spaces constructed in [4] produces examples of non-reflexive Banach spaces of "equal-norm type 2". Then he concluded that these spaces are also of type 2 in the usual sense, by using an unpublished result of G. Pisier asserting that these two notions (i.e. type 2 and equal-norm type 2) are equivalent.

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To make precise the concept of equal-norm type (and also cotype) we introduce the following definition.

DEFINITION. A Banach space X is said to be of *equal-norm type p* or *equal-norm cotype p* if the above definition of type p, respectively cotype p, holds when it is restricted to vectors $\{x_i\}_{i=1}^n$ in X such that

$$\| x_1 \| = \| x_2 \| = \cdots = \| x_n \|.$$

An immediate consequence of [7] and [10] (see also [11]) is that, for every Banach space X,

$$\sup\{p\,;\, X \text{ is of equal-norm type } p\} = \sup\{p\,;\, X \text{ is of type } p\}$$

and a similar equality holds when the word "type" is replaced by "cotype" and "sup" by "inf". Hence, if a Banach space X is, for example, of equal-norm type $p > 1$ then it is also of type $p - \varepsilon$ (in the usual sense) for every $\varepsilon > 0$. It is therefore natural to investigate whether these two notions actually coincide for every p. As we have already pointed out, this is the case if $p = 2$ for both type and cotype.

The aim of this note is to show that, for $1 < p < \infty$; $p \neq 2$, the notions of equal-norm type p or equal-norm cotype p do not necessarily imply those of type p, respectively cotype p.

THEOREM. (i) *For every $1 < p < 2$ there exists a Banach space X (of cotype 2 and with a symmetric basis) which is of equal-norm type p but not of type p in the usual sense.*

(ii) *For every $2 < q < \infty$ there exists a Banach space X (of type 2 and with a symmetric basis) which is of equal-norm cotype q but not of cotype q in the usual sense.*

I. We begin by constructing an auxiliary space. Fix $0 < \gamma < 1$ and $r > 1$, and define a sequence of norms by induction, as follows. For any sequence of real or complex scalars $v = (a_1, a_2, \cdots)$, which is eventually zero, let $\| v \|_0$ denote the norm of v in c_0. Then, for $m \geqq 0$, put

$$\| v \|_{m+1} = \max\left\{ \| v \|_m,\ \gamma \sup\left(\sum_{i=1}^{k} \| P_i v \|_m / k^{1/r'} \right)\right\},$$

where $r' = r/(r-1)$ and the supremum is taken over all the possible decompositions of v into a sequence of mutually disjoint blocks $\{P_i v\}_{i=1}^k$. Since

$$\sum_{i=1}^{k} \| P_i v \|_m / k^{1/r'} \leqq \left(\sum_{i=1}^{k} \| P_i v \|_m^r \right)^{1/r}$$

it follows easily by induction that $\| v \|_m \leqq \| v \|_{l_r}$ for all $m \geqq 0$ and v as above. The expression

$$\| v \| = \lim_{m\to\infty} \| v |_m$$

defines therefore a new norm on the vector space of all sequences which are eventually zero. Its completion $V_{\gamma,r}$ is clearly a Banach space in which the unit vectors form a 1-symmetric basis. Furthermore, the norm in $V_{\gamma,r}$ has the following two obvious properties.

(a) $\| v \| \leqq \| v \|_{l_r}$ for every $v \in l_r$.

(b) If $\{v_i\}_{i=1}^k$ are pairwise disjoint normalized blocks of the unit vector basis in $V_{\gamma,r}$ then

$$\left\| \sum_{i=1}^k v_i \right\| \geqq \gamma k^{1/r}.$$

II. The spaces $V_{\gamma,r}$; $0 < \gamma < 1$, $r > 1$ have the following "universality" property. For given $0 < \gamma < 1$ and $r > 1$, if $D \geqq 1$ is chosen so that $\gamma^{r'D} \leqq 1$ and $U_{D,r}$ is an arbitrary space with a normalized 1-unconditional basis such that every sequence $\{u_i\}_{i=1}^k$; $k \geqq 1$ of normalized pairwise disjoint blocks in $U_{D,r}$ satisfies

$$(*) \qquad D\left\| \sum_{i=1}^k u_i \right\| \geqq k^{1/r}$$

then

$$(**) \qquad \| v \|_{V_{\gamma,r}} \leqq \| v \|_{U_{D,r}},$$

for all $v \in U_{D,r}$. In order to prove this fact, we first consider a sequence $\{w_i\}_{i=1}^k$ of mutually disjoint blocks in $U_{D,r}$. By rearranging them there is no loss of generality in assuming that $\| w_1 \| \geqq \| w_2 \| \geqq \cdots \geqq \| w_k \|$. Then, for any $1 \leqq j \leqq k$, we have

$$D\left\| \sum_{i=1}^k w_i \right\| \geqq D\left\| \sum_{i=1}^j w_i \right\| \geqq \| w_j \| j^{1/r}$$

from which it follows that

$$\sum_{j=1}^k \| w_j \| / k^{1/r'} \leqq D\left\| \sum_{i=1}^k w_i \right\| \left(\sum_{j=1}^k 1/j^{1/r} \right) / k^{1/r'} \leqq r'D\left\| \sum_{i=1}^k w_i \right\|.$$

Suppose now that $\| v \|_m \leqq \| v \|_{U_{D,r}}$, for some integer $m \geqq 0$ and for all $v \in U_{D,r}$. Let $\{P_i v\}_{i=1}^k$ be a decomposition of a vector $v \in U_{D,r}$ into pairwise disjoint blocks. Then, by the inequality established above, we get that

$$\gamma\left(\sum_{i=1}^{k}\|P_i v\|_m\right)/k^{1/r'} \leqq \gamma\left(\sum_{i=1}^{k}\|P_i v\|_{U_{D,r}}\right)/k^{1/r'} \leqq \gamma r' D\left\|\sum_{i=1}^{k} P_i v\right\|_{U_{D,r}}$$
$$= \|v\|_{U_{D,r}},$$

which implies that $\|v\|_{m+1} \leqq \|v\|_{U_{D,r}}$. This proves the inequality (**).

III. The above universality property of $V_{\gamma,r}$ can be used to prove that

$$V_{\gamma,r} \neq l_r \quad \text{(setwise)},$$

for every value of $0<\gamma<1$ and $r>1$. This proof requires the use of the space introduced in [5]. In this article, W. B. Johnson constructed an example of a Banach space T with a normalized 1-unconditional basis $\{e_n\}_{n=1}^{\infty}$ which is not equivalent to the unit vector basis of l_1 but which has the property that among any $2k$ mutually disjoint normalized blocks $\{u_i\}_{i=1}^{2k}$ of $\{e_n\}_{n=1}^{\infty}$ one can find at least k blocks, say $\{u_i\}_{i=1}^{k}$, such that

$$\left\|\sum_{i=1}^{k} a_i u_i\right\| \geqq \delta \sum_{i=1}^{k} |a_i|,$$

for every choice of $\{a_i\}_{i=1}^{k}$, where $\delta=\frac{1}{2}$. By a suitable modification in the construction of T we can obtain a space T_δ satisfying the above inequality for an arbitrary $0<\delta<1$.

We introduce now the following notation. For a vector $u=\sum_{n=1}^{\infty} a_n e_n$ in a space with a 1-unconditional basis $\{e_n\}_{n=1}^{\infty}$ and a positive number α we put (formally) $|u|^{\alpha}=\sum_{n=1}^{\infty}|a_n|^{\alpha} e_n$. Let $T_\delta^{(r)}$ be the r-convexification of T_δ i.e. the completion of the space of all $u=\sum_{n=1}^{\infty} a_n e_n \in T_\delta$ for which also $|u|^r \in T_\delta$, endowed with the norm

$$\|u\|_{T_\delta^{(r)}} = \||u|^r\|_{T_\delta}^{1/r}.$$

The space $T_\delta^{(r)}$ is a r-convex lattice with respect to the coordinatewise order (in general, a Banach lattice X is said to be *r-convex* for some $r>1$ if there exists a constant M so that, for every choice of vectors $\{x_i\}_{i=1}^{m}$ in X, we have

$$\left\|\left(\sum_{i=1}^{m}|x_i|^r\right)^{1/r}\right\| \leqq M\left(\sum_{i=1}^{m}\|x_i\|^r\right)^{1/r}.$$

For additional details on r-convexity we refer the reader to [8] 1-d). It follows directly from the definition of the norm in $T_\delta^{(r)}$ that among any $2k$ mutually disjoint normalized blocks $\{u_i\}_{i=1}^{2k}$ in $T_\delta^{(r)}$ one can choose k blocks, say $\{u_i\}_{i=1}^{k}$, so that

$$\left(\sum_{i=1}^{k}|a_i|^r\right)^{1/r} \geqq \left\|\sum_{i=1}^{k} a_i u_i\right\|_{T_\delta^{(r)}} \geqq \delta^{1/r}\left(\sum_{i=1}^{k}|a_i|^r\right)^{1/r},$$

for every choice of scalars $\{a_i\}_{i=1}^k$. Hence, any set $\{u_i\}_{i=1}^k$ of pairwise disjoint normalized blocks in $T_\delta^{(r)}$ satisfies the condition (*) of II with $D = (4/\delta)^{1/r}$. Consequently, if $V_{\gamma,r} = l_r$ (setwise) for some $0 < \gamma < 1$ and $r > 1$ then, by (**) of II, also $T_\delta^{(r)} = l_r$ (setwise) for any $0 < \delta < 1$ chosen to satisfy $\gamma r'(4/\delta)^{1/r} \leqq 1$. This is however contradictory since, as easily verified, it would imply that T_δ itself coincides with l_1 (for those values of δ as above), contrary to the properties of T_δ established in [5]. Thus, $V_{\gamma,r} \neq l_r$ (setwise).

The author is grateful to G. Pisier for the indirect argument presented above in order to prove that $V_{\gamma,r}$ differs from l_r.

IV. The proof of the Theorem requires in addition the following Proposition which is, in fact, a slight generalization of [6] corollary 7.3.

PROPOSITION. *Let X be a Banach space with a normalized 1-symmetric basis $\{x_n\}_{n=1}^\infty$ which is a 2-convex lattice with respect to the coordinatewise order. There exists a constant $C < \infty$ such that, whenever $\{u_i\}_{i=1}^k$ are arbitrary and $\{v_i\}_{i=1}^k$ pairwise disjoint vectors in X so that, for each $1 \leqq i \leqq k$, u_i and v_i have the same distribution, then*

$$C\int_0^1 \left\| \sum_{i=1}^k r_i(t)u_i \right\| dt \geqq \left\| \sum_{i=1}^k v_i \right\|.$$

In the above statement, two vectors u and v in a space with a 1-symmetric basis $\{x_n\}_{n=1}^\infty$ are considered to have the *same distribution* if, in their expansions with respect to $\{x_n\}_{n=1}^\infty$, both u and v have the same non-zero coefficients each with the same multiplicity.

PROOF OF THE PROPOSITION. We may suppose without loss of generality (cf. [1] or [8] I.d.8) that the 2-convexity constant of X is equal to one. Then the 2-concavification $X_{(2)}$ of X (i.e. the space of all $x \in X$ for which $|x|^{\frac{1}{2}} \in X$, endowed with the norm $\|x\|_{X_{(2)}} = \||x|^{\frac{1}{2}}\|_X^2$) is a Banach space in which the sequence $\{x_n\}_{n=1}^\infty$ is still a normalized 1-symmetric basis.

Let $\{u_i\}_{i=1}^k$ and $\{v_i\}_{i=1}^k$ be two sequences of vectors as in the statement of the Proposition. By Khintchine's inequality

$$\int_0^1 \left\| \sum_{i=1}^k r_i(t)u_i \right\|_X dt \geqq \left\| \int_0^1 \left| \sum_{i=1}^k r_i(t)u_i \right| dt \right\|_X \geqq 2^{-\frac{1}{2}} \left\| \left(\sum_{i=1}^k |u_i|^2 \right) \right\|_X^{1/2}.$$

Hence, in order to prove the Proposition, it would suffice to show that

$$\left\| \left(\sum_{i=1}^k |u_i|^2 \right)^{\frac{1}{2}} \right\|_X \geqq \left\| \sum_{i=1}^k v_i \right\|_X,$$

or, equivalently, that

$$\left\| \left(\sum_{i=1}^{k} |u_i|^2 \right)^{\frac{1}{2}} \right\|_{X_{(2)}} \geqq \left\| \sum_{i=1}^{k} |v_i|^2 \right\|_{X_{(2)}}.$$

This fact however is proved in [6] lemma 7.2 in the context of rearrangement invariant function spaces and extends trivially to the present setting of spaces with a 1-symmetric basis. ■

V. Proof of Part (ii) of the Theorem. Fix $q > 2$ and let X be the 2-convexification of the space $V_{\gamma,r}$ introduced in I, where $r = q/2$ and $0 < \gamma < 1$ is chosen arbitrarily. Clearly, X is a 2-convex space with a 1-symmetric basis. Let $\{u_i\}_{i=1}^k$ be a sequence of normalized vectors in X. Let $\{v_i\}_{i=1}^k$ be mutually disjoint vectors in X built in such a manner that, for each $1 \leqq i \leqq k$, v_i has the same distribution as u_i. It follows easily from (b) of I that

$$\left\| \sum_{i=1}^{k} v_i \right\|_X = \left\| \left| \sum_{i=1}^{k} v_i \right|^2 \right\|_{V_{\gamma,r}}^{1/2} = \left\| \sum_{i=1}^{k} |v_i|^2 \right\|_{V_{\gamma,r}}^{1/2} \geqq (\gamma k^{1/r})^{\frac{1}{2}} = \gamma^{\frac{1}{2}} k^{1/q}$$

and therefore, by the Proposition proved in IV, there is a constant $C < \infty$ (independent of $\{u_i\}_{i=1}^k$) so that

$$C \int_0^\infty \left\| \sum_{i=1}^{k} r_i(t) u_i \right\|_X dt \geqq \gamma^{\frac{1}{2}} k^{1/q},$$

i.e. X is of equal-norm cotype q. On the other hand, if X were of cotype q in the usual sense then, in view of (a) of I, $V_{\gamma,r}$ would coincide setwise with l_r which was shown to be false in III. Since X is 2-convex and of cotype $q + \varepsilon$ for every $\varepsilon > 0$ it follows immediately that X is also of type 2 (cf. [9] or [8] 1.f.17). ■

VI. Proof of Part (i) of the Theorem. The dual X^* of the space X, defined above, has a symmetric basis, is of cotype 2 but it is not of type p for $1/p + 1/q = 1$ (cf. [2], [9] or [8] 1.e.17). Therefore, the proof of Part (i) will be completed if we also show that X^* is of equal-norm type p.

Let $\{w_i^*\}_{i=1}^k$ be a sequence of normalized vectors in X^*. Since X is clearly a superreflexive Banach lattice it follows from [10] remarque 2.10 that

$$Qf = \sum_{n=1}^{\infty} \left(\int_0^1 r_n(t) f(t) dt \right) r_n$$

defines a bounded projection from $L_2(X)$ onto its subspace RADX generated by elements of the form yr_n with $y \in X$ and $n = 1, 2, \cdots$. Hence, there exists a constant $A < \infty$ so that

$$\int_0^1 \left\| \sum_{i=1}^k r_i(t) w_i^* \right\|_{X^*} dt \leqq \left(\int_0^1 \left\| \sum_{i=1}^k r_i(t) w_i^* \right\|_{X^*}^2 dt \right)^{\frac{1}{2}}$$

$$\leqq A \sup \left\{ \sum_{i=1}^k w_i^* w_i ; \{w_i\}_{i=1}^k \subset X \quad \text{and} \quad \int_0^1 \left\| \sum_{i=1}^k r_i(t) w_i \right\|_X^2 dt \leqq 1 \right\}.$$

However, by using the fact that X is of equal-norm cotype q and by applying an argument similar to that presented in the first part of II to an arbitrary sequence $\{w_i\}_{i=1}^k$ in X satisfying $\int_0^1 \| \Sigma_{i=1}^k r_i(t) w_i \|_X^2 dt \leqq 1$, we get

$$\sum_{i=1}^k \| w_i \|_X / k^{1/p} \leqq pC\gamma^{-\frac{1}{2}} \int_0^1 \left\| \sum_{i=1}^k r_i(t) w_i \right\|_X dt \leqq pC\gamma^{-\frac{1}{2}}.$$

Thus

$$\int_0^1 \left\| \sum_{i=1}^k r_i(t) w_i^* \right\|_{X^*} dt \leqq pCA\gamma^{-\frac{1}{2}} k^{1/p},$$

i.e. X^* is indeed of equal-norm type p. ∎

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